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COUPLING CONSTANTS in ASYMPTOTIC EXPANSIONS [1]

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Abstract

Perturbation theory is a powerful tool in manipulating dynamical system. However, it is reliable only for infinitesimal perturbations except the first few approximations. We propose to dispose this problem by means of perturbation group, and find that the coupling constant approaches to zero in the limit of high order perturbations. The Landè factor and the ground state energy of one dimensional φ^4 theory are also concerned.

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1 Introduction

Perturbation theory is widely used in solving dynamical systems, since generally we do not know how to deal with non-linear systems directly [9, 10]. Although this method together with the manipulation of the radiative corrections by renormalization scheme has made prominent success of quantum electrodynamics in the theoretical predictions for Lamb shift and anomalous magnetic moment. However, early in 1952, Dyson [1] argued that all the asymptotic series used in quantum electrodynamics after renormalization in mass and charge are divergent, and that the origin in the complex plane of the coupling constant is a pole. And even earlier, Titchmarsh [2] asserted that the perturbation series are asymptotic series. In 1969, Bender and Wu [3] discussed the φ^4-model in one dimension in their pioneer works. They found that the ground state energy $E_0(\lambda)$ is divergent, and there are an infinite sequence of poles of the resolvent when the phase of the coupling constant goes near to $\pm\frac{3}{2}\pi$, in addition to the cut from the origin to $-\infty$ along the real axis pointed out by Jaffe [4]. In 1976, Lipatov [7] investigated the renormalizable polynomial interaction scalar model. After effecting the Watson-Sommerfeld transformation, he proved the existence of the ultraviolet fixed point of the theory. Brezin et. al. [6] re-derived the results obtained by Bender and Wu [5] in the model with internal $O(n)$ symmetry. In 2002, Kazakov and Popov [8] showed that the asymptotic series of the Gell-Mann-Low β-function cannot be recovered by its first coefficients of the perturbation series and their asymptotic values without invoking additional information. At the same time, the mathematicians also paid great attention to perturbation series. In 1967, Kato [9] set up a theorem for the analytic behavior of of the eigenvalues of teh linear operators in analytic family. Recently, Reed and Simon [10] presented an systematic and extensive review for the theory of perturbation. There the plague of the asymptotic series is illustrated through concrete examples.

Now we propose to dispose this problem by means of perturbation group method, explore the applications of perturbation group in quantum field theory, and show that the coupling constant in the perturbation series should be varied with respect to the order of approximation as Dyson once expected. Finally, in order to show the availability of the perturbation group scheme, we take a glance over the applications of perturbation group in anomalous magnetic moment of electron, Gell-Mann-Low β-function and the ground state energy of one dimensional φ^4-model.

2 Perturbation Group

It is well-known that the Rayleigh-Schrödinger series is only the asymptotic series of the solution for the Schrodinger equation. The series $\sum_{n=0}^{N} a_n\lambda^n$ is referred as an asymptotic to $f(\lambda)$ means that

$$\lim_{\lambda\downarrow 0}\left(f(\lambda)-\sum_{n=0}^{N} a_n\lambda^n\right)\Big/\lambda^N=0 \tag{1}$$

It is shown that if the asymptotic series is not convergent, then it is reliable only when λ is in the neighbor of zero [9, 3, 10], otherwise it gives no reliable information for the values of the function $f(\lambda)$ at some finite value of λ as one might expect. Therefore the perturbation series is reliable only for infinitesimal perturbations rather than any finite perturbation except the first few approximations. In view of improving the exactness and reliableness of the perturbation method , it is preferable to separate the perturbation into many steps and limit the perturbations to be infinitesimal in each step.

Let H_0 and K be two elements in a Banach space $\mathcal{B}$, and $\beta\in\mathbb{Z}$, then

$$H=H_0+\beta K\in\mathcal{B}.$$

Let $T(\beta,\beta_0)$ be a translation in $\mathcal{B}$,

$$T(\beta,\beta_0):\qquad H_{\beta_0}=H_0+\beta_0 K\longmapsto H_\beta=H_0+\beta K. \tag{2}$$

Let the representation $U(\beta,\beta_0)$ of the translation $T(\beta,\beta_0)$ be a transformation on a Hilbert space $\mathcal{H}$ defined by $U(\beta_1,\beta_0)\in\mathcal{L}(\mathcal{H})$,

$$U(\beta_1,\beta_0)\varphi_{\beta_0}=\varphi_{\beta_1} \tag{3}$$

where φ_{β_0} and φ_{β_1} are the eigenvectors of H_{β_0} and H_{β_1} respectively. And it is assumed that when β varies continuously from β_0 to β_1, the eigenvector of $H(\beta,\beta_0))$ varies continuously from $\varphi(\beta_0)$ to $\varphi(\beta_1)$. Since

$$U(\beta_2,\beta_1)U(\beta_1,\beta_0)\varphi_{\beta_0}=U(\beta_2,\beta_1)\varphi_{\beta_1}=\varphi_{\beta_2}, \tag{4}$$

and

$$U(\beta_2,\beta_0)\varphi_{\beta_0}=\varphi_{\beta_2}, \tag{5}$$

Thus

$$U(\beta_2, \beta_1)U(\beta_1, \beta_0) = U(\beta_2, \beta_0). \tag{6}$$

Besides

$$U(\beta, \beta) = 1, \tag{7}$$

and

$$U(\beta_1, \beta_0)^{-1} = U(\beta_0, \beta_1). \tag{8}$$

Therefore the transformations $U(\beta_1, \beta_0)$ form a group.

3 Generator for Perturbation Group

Let us first setup the perturbation equation for the perturbation group. The derivative of $U(\beta, \beta_0)$ with respect to β is given as follows,

$$\begin{aligned}
\frac{\partial U(\beta, \beta_0)}{\partial \beta} &= \lim_{\Delta\beta \to 0} \frac{U(\beta + \Delta\beta, \beta_0) - U(\beta, \beta_0)}{\Delta\beta} \\
&= \lim_{\Delta\beta \to 0} \frac{U(\beta + \Delta\beta, \beta) - U(\beta, \beta)}{\Delta\beta} U(\beta, \beta_0) \\
&= \left. \frac{\partial U(\beta', \beta)}{\partial \beta'} \right|_{\beta' = \beta} U(\beta, \beta_0).
\end{aligned} \tag{9}$$

where we denote

$$G(\beta) \equiv \left. \frac{\partial U(\beta', \beta)}{\partial \beta'} \right|_{\beta' = \beta} \tag{10}$$

as the generator of the perturbation group. thus we have the perturbation equation for the perturbation group as follows,

$$\frac{\partial U(\beta, \beta_0)}{\partial \beta} = G(\beta)U(\beta, \beta_0). \tag{11}$$

The infinitesimal transformation can be written as

$$U(\beta + \Delta\beta, \beta) = \exp\{G(\beta)\Delta\beta\}, \quad \text{for} \quad \Delta\beta \to 0 \tag{12}$$

Therefore we can proceed along with the infinitesimal perturbations step by step, and finally obtain the perturbation with finite coupling constant. Let C be a section of line connecting two points β_0 and β_1 in

the analytic region of the complex plane of β, while the line segment is divided into n sections by $n-1$ points $\beta(s_k), s_k \in \mathbb{R}, (k = 1, 2, 3, ..., n-1)$ on the line,

$$\beta(s_k) = \beta(s_0) + k\Delta\beta, \quad (k \in \mathbb{I}) \qquad \beta(s_n) = \beta(s_f). \tag{13}$$

Then

$$\begin{aligned}
&U(\beta_f, \beta_0) = \lim_{n\to\infty} U(\beta_0 + n\Delta\beta, \beta_0 + (n-1)\Delta\beta)....U(\beta_0 + k\Delta\beta, \beta_0 + (k-1)\beta))...U(\beta_0 + \Delta\beta, \beta_0) \\
&= \lim_{n\to\infty} \mathbb{P}\exp\{G(\beta_0 + (n-1)\Delta\beta)\Delta\beta\}... \exp\{G(\beta_0)\Delta\beta\} \\
&= \mathbb{P}\exp\{\lim_{n\to\infty} \sum_{k=0}^{n-1} G(\beta + k\Delta\beta)\Delta\beta)\} \\
&= \mathbb{P}\exp\Big\{ \int_{\beta_0}^{\beta_f} G(\beta)d\beta \Big\} \\
&= 1 + \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\beta_0}^{\beta_f} d\beta_1 ... \int_{\beta_0}^{\beta_f} d\beta_n \mathbb{P}G(\beta_1)G(\beta_2)...G(\beta_n).
\end{aligned} \tag{14}$$

where $\mathbb{P}$ denotes the perturbation ordered product by the coupling constant $\beta \in \mathbb{R}$,

$$\mathbb{P}G(\beta(s_2))G(\beta(s_1)) = \begin{cases} G(\beta(s_2))G(\beta(s_1)), & \text{for} s_2 > s_1, \\ G(\beta(s_1))G(\beta(s_2)), & \text{for} s_1 > s_2, \end{cases} \tag{15}$$

and use is made of the fact that

$$\mathbb{P}\int_{\beta_0}^{\beta_f} d\beta_1 ... \int_{\beta_0}^{\beta_{n-1}} d\beta_n G(\beta_1)G(\beta_2)...G(\beta_n) = \frac{1}{n!}\int_{\beta_0}^{\beta_f} d\beta_1 ... \int_{\beta_0}^{\beta_f} d\beta_n \mathbb{P}G(\beta_1)G(\beta_2)...G(\beta_n). \tag{16}$$

Note that in view of the of the perturbation ordered product, there are no commutators between the generators in the exponential in contrast to the Campbell-Baker-Hausdorff formula.

The transformation $U(\beta_f, \beta_0)$ in eq.(14) can be manipulated through iteration, i. e., insert the zeroth approximation into the right hand side integrals and obtain the first approximation, then iterate. In time dependent quantum mechanics, the perturbation transformation U and thus the generators are also time dependent.

4 Perturbation Group in Quantum Field Theory

The evolution of the state vector $\varphi_I(\beta; t)$ in the interaction picture is effected by the evolution operator $U(\beta, \beta_0; t_1, t_0)$,

$$i\frac{\partial U(\beta_0 + \Delta\beta, \beta_0; t, t_0)}{\partial t} \qquad = \Delta\beta H_I(t) U(\beta_0 + \Delta\beta, \beta_0; t, t_0), \tag{17}$$

where $H_I(t) \equiv \int d^3x \mathcal{H}(t, \mathbf{x})$ is the Hamiltonian 3-density, and

$$U(\beta_0 + \Delta\beta, \beta_0; t, t_0) = \mathbb{T} \exp\{i \int dt \Delta\beta H_I(t)\} \tag{18}$$

It can be easily seen that the perturbation transformations form a group. While the generator for the infinitesimal perturbation transformation is

$$G(\beta) = i \int dt H_I(t), \tag{19}$$

Then the perturbation group equation is

$$\frac{\partial U(\beta_1, \beta_0; t, t_0)}{\partial \beta} = \Big\{ \int dt H_I(t) \Big\} U(\beta_1, \beta_0; t, t_0) \tag{20}$$

Therefore the perturbation transformation $U(\beta, 0; t, t_0)$ can be obtained as follows,

$$\begin{aligned} U(\beta, 0; t, t_0) \qquad &= \mathbb{P}\mathbb{T} \exp\left\{ -i \int\limits_0^\beta d\beta' \int_{t_0}^t dt H_I(t') \right\} \\ &= 1 + \sum_n \frac{1}{n!} \frac{(-i)^n}{n!} \beta^n \int\limits_{t_0}^t dt'_1 ... \int\limits_{t_0}^t dt'_n \mathbb{T} H_I(t'_1) ... H_I(t'_n). \end{aligned} \tag{21}$$

In view of the reducing factor $\frac{1}{n!}$ before the n-th order term in the series, it seems that the contributions of the higher corrections are overestimated in the conventional series.[8]. This reducing factor $\frac{1}{n!}$ can also be absorbed into the coupling constant, and define the coupling constant as a function $g(n)$ of order of approximation n,

$$g(n) \equiv \frac{g}{(n!)^{1/n}} \approx \frac{e}{n} g, \tag{22}$$

for large n, where e is the constant of the natural logarithm. Therefore

$$\lim_{n \to \infty} g(n) = 0, \tag{23}$$

as Dyson once expected [1]. Then the series recovers its conventional form.

5 Quantum electrodynamics, quantum chronodynamics and φ^4 theory

It is well-known that quantum electrodynamics achieved great success in theoretical prediction for anomalous magnetic moment of electron[11]. According to Furry's theorem, it seems to be more reasonable to choose the coupling constant $\alpha \equiv e^2$ as the parameter in loop expansions.

The Landè factor g for anomalous magnetic moment of pure quantum electrodynamics in perturbation group scheme [11] is

$$g = 2\Big[1 + C_1\Big(\frac{\alpha}{\pi}\Big) + C_2\frac{1}{2!}\Big(\frac{\alpha}{\pi}\Big)^2 + C_3\frac{1}{3!}\Big(\frac{\alpha}{\pi}\Big)^3 + C_4\frac{1}{4!}\Big(\frac{\alpha}{\pi}\Big)^4 + ...\Big], \tag{24}$$

where the coefficients C_i's are obtained in renormalization.

Taking the contributions from QED into account only, the g-factor obtained in perturbation group approach is

$$g_{\rm QED}^{\rm PG} = 2(1 + 0.0011605260402..), \tag{25}$$

while recent experimental data is [11]

$$g_{\rm exp} = 2(1 + 0.0011596521884...). \tag{26}$$

It is readily seen that $g_{\rm QED}^{\rm PG}$ is a little bit closer to $g_{\rm exp}$ than $g_{\rm QED}^{\rm ord} = 0.00116584705(18)$ obtained in the ordinary approach [11] up to the fourth order of α. Besides, in ordinary manipulations one might still worry about the higher terms may spoil the convergence of the asymptotic series [1], while the situation in the perturbation group scheme is substantially improved.

In quantum chronodynamics, the Gell-Mann-Low β-function in the perturbation scheme can be defined by the following series [12],

$$\beta(g) = -\beta_0 g^3 - \beta_1\frac{1}{2!}g^5 - \beta_2\frac{1}{3!}g^7 + O(g^9), \tag{27}$$

Thus it is evident that the behavior of the ordinary β-function is substantially modified by the factors $1/n!$ in the perturbation group scheme. And this is also true for the ground state energy $E_0(\lambda)$ of one dimensional φ^4-model [3].

Actually, the Hamiltonian of the one-dimensional anharmonic oscillator is

$$H = \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}m^2\varphi^2 + \lambda\varphi^4. \tag{28}$$

Then the ground state energy of the oscillator as observed by Bender and Wu [5] is given by The ground state energy of the anharmonic oscillator is

$$E_0(\lambda) = \frac{1}{2}m + \sum_{n=1}^{\infty} mA_n(\lambda/m^3)^n. \tag{29}$$

The detail asymptotic growth of A_n is

$$A_n \sim (-1)^{n+1}(6/\pi^3)^{1/2}\Gamma(n+\frac{1}{2})3^n. \tag{30}$$

In using the Stirling's formula, the perturbation group scheme gives out the asymptotic values of coefficients A_n as follows,

$$\begin{aligned} A_n \sim &\quad (-1)^{n+1}\left(\frac{6}{\pi^3}\right)^{1/2}\frac{\Gamma(n+\frac{1}{2})3^n}{n!} &(31)\\ \sim &\quad (-1)^{n+1}\left(\frac{6}{\pi^3}\right)^{1/2}\frac{2e3^n}{(n+1)^{3/2}} &(32)\end{aligned}$$

where e is the constant for natural logarithm. It can be readily seen that when $m = 1$ and $0 < \lambda < \frac{1}{3}$, and E_0 will approach to a finite value when $n \to \infty$. When $\lambda = 0.2$, $E_0 \approx 1.131568$.

6 Conclusion

We propose the perturbation group and perturbation equation. Using the Rayleigh-Schrodinger series, we derive the generator for perturbation transformations, give out the transformation for finite perturbations, and find that the coupling constant in quantum field theory is varied and approaches to zero as the order of approximation goes to infinity in ordinary perturbation series as Dyson once expected. It is shown that the prediction for Landè factor in perturbation scheme matches the experiments better than that obtained in ordinary manipulations, and the behaviors of the asymptotic series in quantum field theories are substantially modified.

It is worthy to mention that perturbation group is different from renormaliztion group. In renormaliztion group, the coupling constant is related to the energy scale of the renormalization point. But in perturbation group, the coupling constant is referred as an independent variable. Actually the final limits of the coupling constant variables in perturbation group transformations are just the running coupling constants in the perturbation series in physics. If one manipulates only by means of renormaliztion group, then the asymptotic series in quantum field theory are in general divergent even after renormalization whenever the perturbations are non-infinitesimal. While the behaviors of the asymptotic series are substantially modified in perturbation group scheme.

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